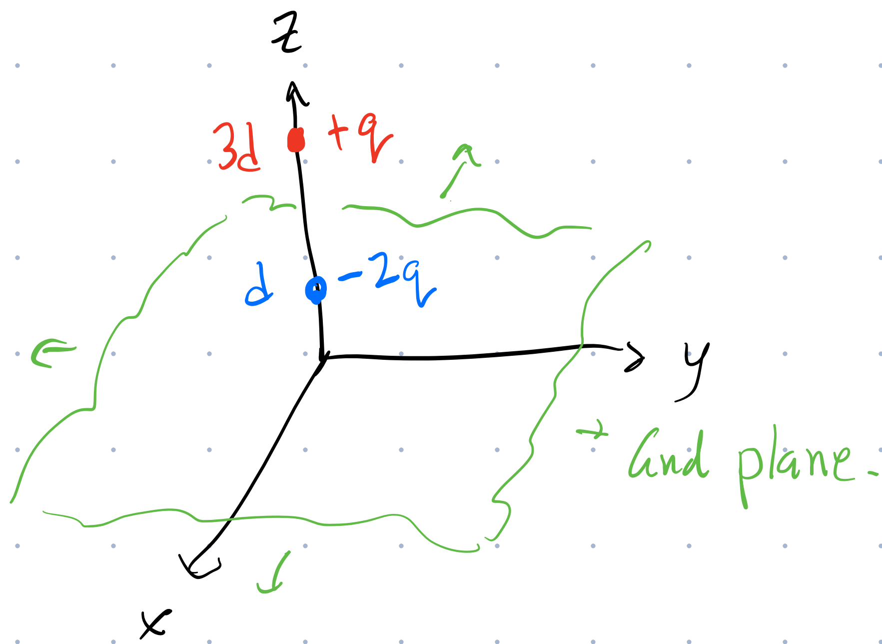
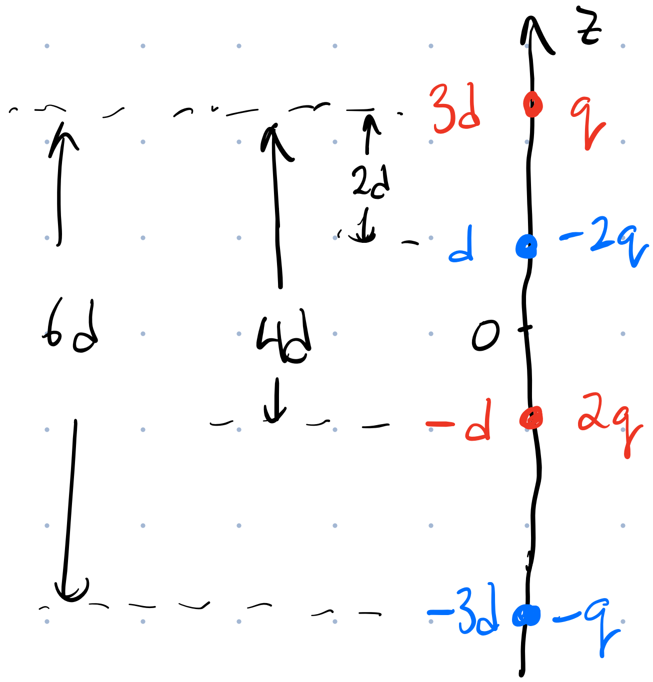
PHYS 301 Tutorial #6 Sol'ns Oct. 25, 2024

1.



By the method of images, this config. is equiv. to



Force on +q charge @ 3d.

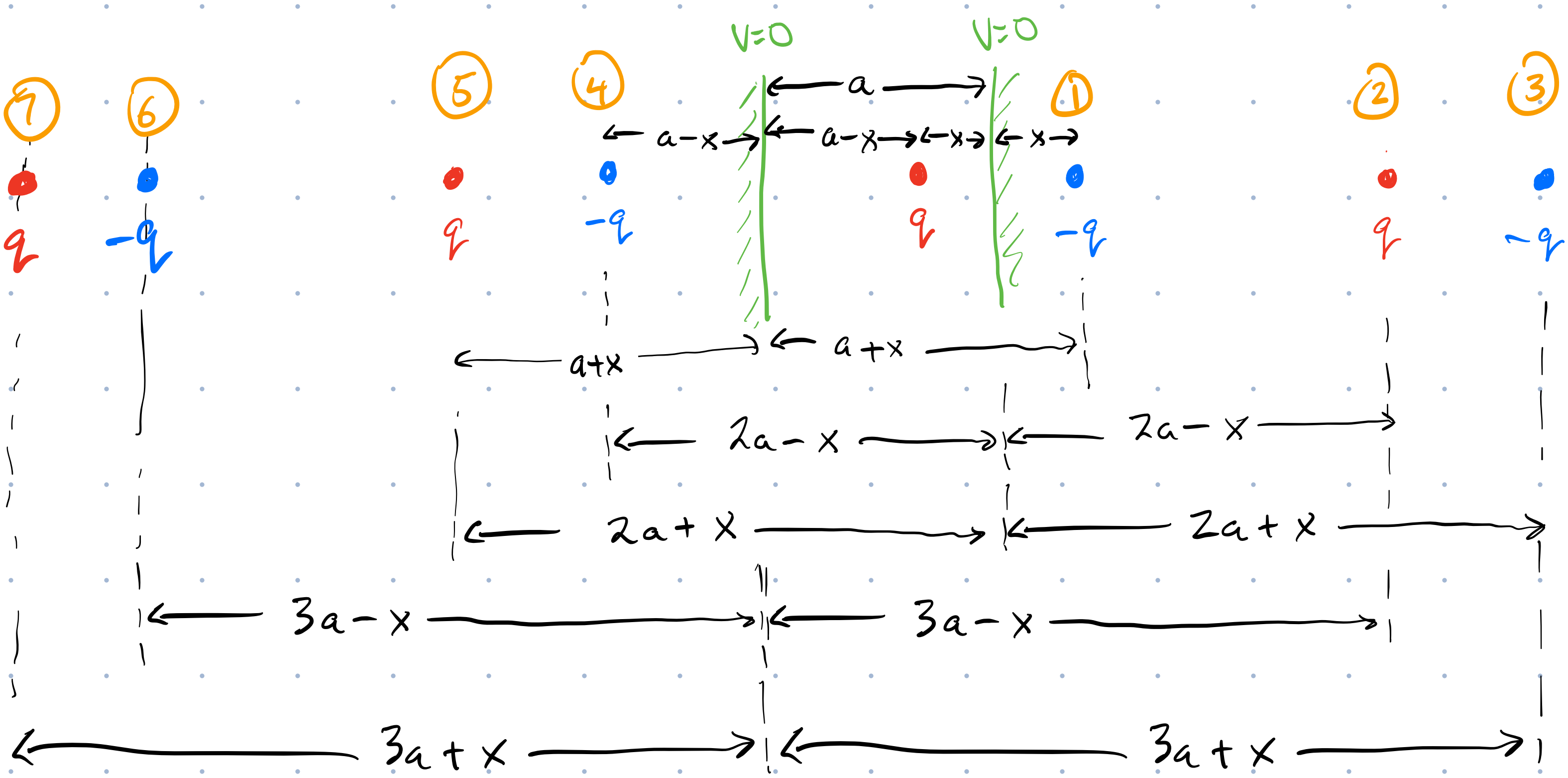
$$\vec{F} = \frac{q^2}{4\pi\varepsilon_0 d^2}\left[-\frac{2}{2^2}+\frac{2}{4^2}-\frac{1}{6^2}\right]\hat{z}$$

$$\vec{F} = \frac{q^2}{4\pi\varepsilon_0 d^2}\left[-\frac{1}{2}+\frac{1}{8}-\frac{1}{36}\right]\hat{z}$$

$$= \frac{q^2}{4\pi\varepsilon_0 d^2} \frac{-36+9-2}{72}\hat{z} = \frac{q^2}{4\pi\varepsilon_0 d^2}\frac{-29}{72}\hat{z}$$

$$\boxed{\vec{F} = \frac{-29 q^2}{288\pi\varepsilon_0 d^2}\hat{z}}$$

2. (a) Get images of image charges! $\longrightarrow \hat{x}$



Force on q: $\vec{F}_1 = \frac{k_e q^2}{4x^2}\hat{x}$ $\qquad k_e = \frac{1}{4\pi\varepsilon_0}$

$$\cancel{\vec{F}_2 = -\frac{k_e q^2}{4a^2}\hat{x}}$$

$$\vec{F}_3 = \frac{k_e q^2}{4(a+x)^2}\hat{x} \qquad \vec{F}_4 = -\frac{k_e q^2}{4(a-x)^2}\hat{x} \qquad \cancel{\vec{F}_5 = \frac{k_e q^2}{4a^2}\hat{x}}$$

$$\vec{F}_6 = \frac{-k_e q^2}{(4a-2x)^2}\hat{x} = -\frac{k_e q^2}{4(2a-x)^2} \qquad \cancel{\vec{F}_7 = \frac{k_e q^2}{(4a)^2}\hat{x}}$$

will cancel w/ + image charge to right of 3

$\vdots$

The positive image charges are always $\pm 2a, \pm 3a, \pm 4a, \pm 5a, \ldots$ away from the real charge $+q$. Force due to these positive charges cancel.

To the right of q, the $-$ image charges are a dist.

$$2x,\ 2a+2x,\ 4a+2x,\ 6a+2x, \ldots$$

away & pull q right ($+\hat{x}$-dir'n)

To the left of q, the $-$ image charges are a dist

$$2a-2x,\ 4a-2x,\ 6a-2x, \ldots$$

away & pull q to left ($-\hat{x}$-dir'n)

$$\therefore \vec{F}_{net} = k_e q^2 \left[\frac{1}{4x^2} + \frac{1}{4(a+x)^2} + \frac{1}{4(2a+x)^2} + \cdots \right.$$

$$\left. - \frac{1}{4(a-x)^2} - \frac{1}{4(2a-x)^2} - \frac{1}{4(3a-x)^2} - \cdots \right] \hat{x}$$

$$\boxed{= \frac{k_e q^2}{4} \left[\frac{1}{x^2} + \frac{1}{(a+x)^2} + \frac{1}{(2a+x)^2} + \cdots - \frac{1}{(a-x)^2} - \frac{1}{(2a-x)^2} - \frac{1}{(3a-x)^2} - \cdots \right]}$$

(b) If $a \to \infty$, get q a dist. x from a single gnd plane. In that case, expect

$$\vec{F} = \frac{k_e q^2}{4x^2} \hat{x}$$

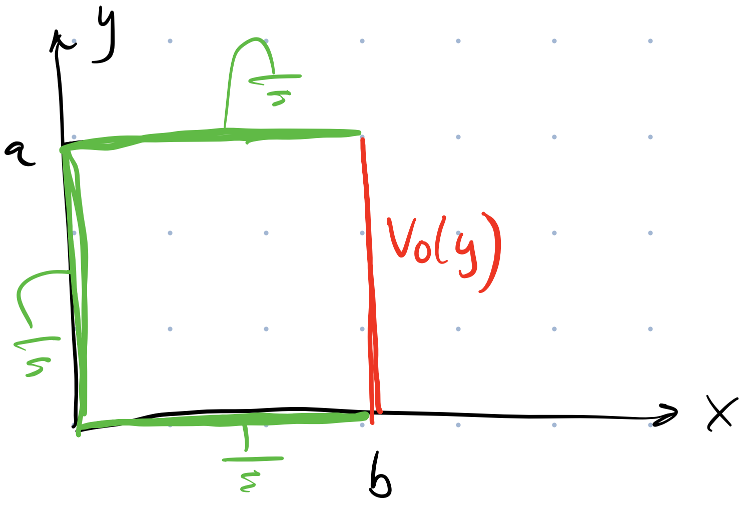
In our sol'n for (a), only the $\frac{1}{x^2}$ term survives as $a \to \infty$ ✓

(c) if $x = a/2$, expect $\vec{F} = 0$.

Here, our sol'n to (a) becomes:

$$\vec{F}_{net} = \frac{k_e q^2}{4}\left[\cancel{\frac{1}{(a/2)^2}} + \cancel{\frac{1}{(3a/2)^2}} + \cancel{\frac{1}{(5a/2)^2}} + \ldots - \cancel{\frac{1}{(a/2)^2}} - \cancel{\frac{1}{(3a/2)^2}} - \cancel{\frac{1}{(5a/2)^2}} - \ldots \right] = 0 \checkmark$$

3. z-axis out of screen/page.



Use separation of variables w/ $V = V(x,y)$ (no z-dependence).

$$\nabla^2 V = \nabla^2 X(x) Y(y) = 0$$

$$Y \frac{d^2X}{dx^2} + X \frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = k^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2$$

$$\Rightarrow X = Ae^{kx} + Be^{-kx}$$
$$Y = C\sin ky + D\cos ky$$

$\therefore$ In general, $V = XY = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$

b.c.s (i) $V=0$ @ $x=0$

$$\Rightarrow (A+B)(C\sin ky + D\cos ky) = 0$$

$$\therefore A+B=0 \quad \boxed{B=-A}$$

(ii) $V=0$ @ $y=0$

$$\underbrace{A(e^{kx}-e^{-kx})}(D) = 0 \quad \therefore \boxed{D=0}$$

$2\sinh kx$

$$\therefore V = E \sinh kx \sin ky$$

(iii) $V=0$ @ $y=a$

$$E \sinh kx \sin ka = 0$$

$$\therefore ka = n\pi \Rightarrow \boxed{k = \frac{n\pi}{a}}$$

$$\therefore V_n = E \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

Most general sol'n is a superposition of V_n.

$$\boxed{\therefore V(x,y) = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)}$$

$n=0$ doesn't work b/c then $V=0$ everywhere

Need form of $V_0(y)$ @ $x=b$ to go further.

(b) if $V_0(y) = V_0$ (indep. of y) when $x=b$

then $$V_0 = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

To find E_m, mult. by $\sin\left(\frac{m\pi y}{a}\right)$ and integrate w.r.t. y from $y=0$ to a

$$\underbrace{V_0 \int_{y=0}^{a} \sin\left(\frac{m\pi y}{a}\right) dy}_{} = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi b}{a}\right) \underbrace{\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy}_{}$$

$$-\frac{a}{m\pi} \cos\left(\frac{m\pi y}{a}\right)\Bigg|_0^a = -\frac{a}{m\pi}\left[\cos(m\pi) - 1\right]$$
$$= \frac{a}{m\pi}\left[1 - \cos(m\pi)\right]$$

zero for $n \neq m$ b/c sine fcns are orthogonal

when $n = m$

$$\int_0^a \sin^2\left(\frac{n\pi y}{a}\right) dy = \frac{1}{2}\int_0^a \left[1 - \cos\left(\frac{2n\pi y}{a}\right)\right] dy$$
$$= \frac{1}{2}\left[y - \frac{a}{2n\pi}\sin\left(\frac{2n\pi y}{a}\right)\right]\Bigg|_0^a$$

$$= a/2$$

$$\therefore \int_{-\pi}^{\pi} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy = \frac{a}{2}\delta_{nm}$$

$$\frac{V_0 \cancel{a}}{m\pi}\underbrace{\left[1-\cos(m\pi)\right]} = \frac{\cancel{a}}{2}\sum_{n=1}^{\infty} E_n \sinh\left(\frac{n\pi b}{a}\right)\delta_{nm}$$

$$= \begin{cases} 0 & m \text{ even} \\ 2 & m \text{ odd} \end{cases} \qquad = \frac{\cancel{a}}{2} E_m \sinh\left(\frac{m\pi b}{a}\right)$$

$$\boxed{\therefore E_m = \frac{4V_0}{m\pi \sinh\left(\frac{m\pi b}{a}\right)} \quad m \text{ odd}}$$

Combining everything, we get:

$$\boxed{V(x,y) = \sum_{n \text{ odd}}^{\infty} \frac{4V_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)}$$

4. From class notes, $V(r,\theta)$ for a spherical shell w/ azimuthal symmetry:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

In the current problem, $V_0 = k\cos 3\theta$ @ $r=R$.
(i.e. on the shell).

<u>Inside</u> V finite @ $r=0$ $\therefore B_l = 0$.

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

@ $r=R$

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

$\cos 3\theta$: There is a triple-angle identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Consider the first few Legendre Polynomials

$$P_0 = 1 \qquad P_1 = x \qquad P_2 = \frac{1}{2}(3x^2 - 1)$$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

Want to construct

$$aP_0 + bP_1 + cP_2 + dP_3 = 4x^3 - 3x$$

no const term $\therefore a = 0$

no x^2 term $\therefore c = 0$

$$bP_1 + dP_3 = 4x^3 - 3x$$

$$bx + \frac{d}{2}(5x^3 - 3x) = 4x^3 - 3x$$

$$\frac{5d}{2}x^3 + \left(b - \frac{3d}{2}\right)x = 4x^3 - 3x$$

$$\therefore \frac{5d}{2} = 4 \Rightarrow d = \frac{8}{5}$$

$$b - \frac{3}{2}\frac{8}{5} = -3$$

$$b - \frac{12}{5} = -3 \qquad \therefore b = \frac{-15 + 12}{5} = -\frac{3}{5}$$

$\therefore$

$$V_0 = k\cos 3\theta = k\left[-\frac{3}{5}P_1(\cos\theta) + \frac{8}{5}P_3(\cos\theta)\right]$$

$$= \frac{k}{5}\left[-3P_1(\cos\theta) + 8P_3(\cos\theta)\right]$$

$$\therefore V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \frac{k}{5}\left[-3P_1(\cos\theta) + 8P_3(\cos\theta)\right]$$

$\therefore$ only $l=1$ & $l=3$ terms have non-zero A_l

$$A_1 R\, P_1(\cos\theta) + A_3 R^3 P_3(\cos\theta) = -\frac{3k}{5}P_1(\cos\theta) + \frac{8k}{5}P_3(\cos\theta)$$

$$\therefore A_1 R = \frac{-3k}{5} \qquad \boxed{A_1 = \frac{-3k}{5R}}$$

$$A_3 R^3 = \frac{8k}{5} \qquad \boxed{A_3 = \frac{8k}{5R^3}}$$

$$\therefore V(r,\theta) = A_1 r P_1(\cos\theta) + A_3 r^3 P_3(\cos\theta)$$

$$= \frac{-3k}{5R} r\cos\theta + \frac{3k}{5R^3} r^3 \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$\boxed{\therefore V(r,\theta) = \frac{k}{5}\left(\frac{r}{R}\right)\cos\theta\left[-3 + 4\left(\frac{r}{R}\right)^2(5\cos^2\theta - 3)\right]}$$